PROBABILISTIC METHODS IN PARTIAL DIFFERENTIAL EQUATIONS

BY

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ABSTRACT

Results on the asymptotic behavior of solutions of the Cauchy problem $\partial u/\partial t = Lu$ as $t \to \infty$ are stated, both for nondegenerate and degenerate elliptic second order operator L. The Dirichlet problem for degenerate L is also studied. The methods used depend on a detailed study of the behavior of solutions of stochastic differential equations.

This work is concerned, for the most part, with the asymptotic behavior of solutions of the Cauchy problem for second order (possibly degenerate) parabolic equations. The methods used are based on recent results in stochastic differential equations. The Dirichlet problem for degenerate elliptic equations is treated briefly by the same methods.

The results of Section 1 are due to Friedman [2]; the results of Sections 2, 3 are due to Friedman and Pinsky [3], [4]. Finally, a detailed account of the results briefly described in Section 4 will appear in a forthcoming paper by Friedman and Pinsky [5].

1. Comparison with the heat equations

Consider the Cauchy problem

$$(1.1)\quad \frac{\partial u}{\partial t} = Lu \equiv \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} \quad (t > 0, x \in R^m).$$

$$(1.2)\qquad u(0,x) = f(x) \quad (x \in R^m).$$

We wish to compare the solution u with the solution v of the heat equation

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$$\frac{\partial v}{\partial t} = \frac{1}{2} \sum_{i=1}^{m} \frac{\partial^2 v}{\partial x_i^2} \quad (t > 0, x \in R^m)$$

subject to the same initial condition (1.2). We assume:

(A_1) $(a_{ij}(x))$ is a positive definite matrix for each x, and $a_{ij}(x)$, $b_i(x)$ are uniformly Hölder continuous and locally Lipschitz continuous. This implies that there is an $m \times m$ matrix $\sigma(x) = (\sigma_{ij}(x))$ satisfying $\sigma\sigma^* = (a_{ij})$ (σ^* = transpose of σ) which is locally Lipschitz continuous.

(A_2) There is a constant matrix $\bar{\sigma} = (\bar{\sigma}_{ij})$ such that

$$|\sigma_{ij}(x) - \bar{\sigma}_{ij}| \leqq \frac{C}{(1 + |x|)^\delta},$$

$$|b_i(x)| \leqq \frac{C}{(1 + |x|)^{1+\delta}}$$

for some $0 < \delta < 1$, $C > 0$.

(A_3) For all x, $\bar{x}$ in R^m,

(1.3) $$|f(x) - f(\bar{x})| \leqq C_0 |x - \bar{x}|^\gamma \quad (0 < \gamma \leqq 2, C_0 > 0).$$

If f is any continuous function bounded by $C_\varepsilon \exp[\varepsilon |x|^2]$ for any $\varepsilon > 0$ (C_ε is a constant), then under very general assumptions (weaker than (A_1),(A_2)) there exists a unique solution of (1.1), (1.2) bounded in each strip $0 \leqq t \leqq T$ by $C' \exp[c\,|x|^2]$ (C', c positive constants depending on T); see for instance [1]. When (1.3) holds then

$$|v(t,x)| \leqq C_1(1 + t + |x|^2)^{\gamma/2} \quad (C_1 \text{ constant});$$

this bound is sharp, i.e., the reverse inequality holds when $f(x) = (1 + |x|^2)^{\gamma/2}$, with a different positive constant C_1.

THEOREM 1. *Suppose* (A_1)–(A_2) *hold with* $\bar{\sigma}_{ij} = \delta_{ij}$ *and* $m \geqq 2$. *Then*

(1.4) $$|u(t,x) - v(t,x)| \leqq C(1 + t + |x|^2)^{(1-\delta)\gamma/2}$$

where C *is a constant.*

When $m = 1$, set $\sigma = \sigma_{11}$, $\bar{\sigma} = \bar{\sigma}_{11}$, $b = b_1$, and assume:

(A_2') $\bar{\sigma} \neq 0$ and

$$|\sigma(x) - \bar{\sigma}| \leqq \frac{C}{(1 + |x|^\delta}, \quad \left| \frac{b(x)}{\sigma(x)} - \tfrac{1}{2}\sigma'(x) \right| \leqq \frac{C}{(1 + |x|)^{1+\delta}}$$

for some $0 < \delta < 1$, $C > 0$.

THEOREM 2. *Suppose* $m = 1$, $\bar{\sigma} = 1$ *and* (A_1), (A_2'), (A_3) *hold. Then the assertion* (1.4) *is valid provided*

$$\int_{-\infty}^{\infty} \left[\frac{b(x)}{\sigma(x)} - \tfrac{1}{2}\sigma'(x)\right] dx = 0. \tag{1.5}$$

Without the assumption (1.5), the assertion (1.4) is generally false even when $\sigma(x) \equiv 1$ and $b(x)$ has compact support.

For the proofs, consider the stochastic differential system

$$d\xi(t) = b(\xi(t))dt + \sigma(\xi(t))dw(t) \tag{1.6}$$

where $w(t)$ is m-dimensional Brownian motion (see [6] for the relevant theory). The following estimate is proved by Friedman [2]:

$$E\left|\xi(t) - \bar{\sigma}w(t)\right|^2 \leqq C[1 + E\left|\xi(0)\right|^2 + t]^{1-\delta}. \tag{1.7}$$

Now (see [6]),

$$u(t,x) = Ef(\xi_x(t))$$
$$v(t,x) = Ef(x + w(t))$$

where $\xi_x(t)$ is the solution of (1.6) with $\xi(0) = x$. Using (1.7), (1.3) and Hölder's inequality, (1.4) follows.

The derivation of (1.7) is rather lengthy; it is based on Ito's calculus and on the construction of comparison functions.

EXTENSIONS.

1) If $m \geqq 3$ and $\delta > 1$ in the condition (A_2), then the assertion (1.4) can be replaced by the stronger assertion:

$$\left|u(t,x) - v(t,x)\right| \leqq C.$$

2) If $0 < \delta < \frac{1}{2}$ then

$$E\left|\xi(t) - \bar{\sigma}w(t)\right|^4 \leqq C[1 + E\left|\xi(0)\right|^4 + t^2]^{1-\delta}. \tag{1.8}$$

Indeed, this can be proved by a straightforward extension of the proof of (1.7). The estimate (1.8) yields extension of Theorems 1, 2 (with the same assertion (1.4)) to the case where $2 < \gamma \leqq 4$, provided $0 < \delta < \frac{1}{2}$. Similarly one can deal with the case $\gamma > 4$.

3) The estimate (1.7) holds also in case $\bar{\sigma}$ is degenerate, provided $\bar{\sigma}\bar{\sigma}^*$ has at least two positive eigenvalues. This leads to a corresponding extension of Theorem 1 in case $\bar{\sigma}\bar{\sigma}^* = (\bar{a}_{ij})$ where $\bar{a}_{ij} = 1$ if $i = j = 1, 2, \cdots, d$ and $\bar{a}_{ij} = 0$ otherwise; $d \geqq 2$.

4) The estimate (1.7) was proved (in [2]) for general equations

$$d\xi(t) = b(t,\xi(t))dt + \sigma(t,\xi(t))dw(t) \tag{1.9}$$

where $b(t,x)$, $\sigma(t,x)$ satisfy the same bounds as in (A_2), uniformly with respect to t. This leads to estimates for the Cauchy problem with time-dependent coefficients.

5) The method of proving (1.7) can be extended to other equations. For example, if for some constant vector λ, the functions

$$\hat{\sigma}_{ij}(t,x) = \sigma_{ij}(t,x+\lambda t)$$

$$\hat{b}_i(t,x) = b_i(t,x+\lambda t) - \lambda$$

satisfy the same conditions as in (A_2), then

$$E\,|\,\xi(t) - \lambda t - \bar{\sigma}w(t)\,|^2 \leqq C[1 + E\,|\,\xi(0)\,|^2 + t]^{1-\delta}.$$

This leads to further estimates for the Cauchy problem.

2. Degenerate parabolic equations

In case $(a_{ij}(x))$ is degenerate in R^m, entirely different asymptotic behavior may occur. Suppose there are a finite number of disjoint sets $G_1,\cdots,G_k$ in R^m, of which $G_1,\cdots,G_{k_0}$ consist of points $z_1,\cdots,z_{k_0}$, and the G_j $(k_0+1 \leqq j \leqq k)$ are closed bounded domains with C^3 boundary ∂G_j. Assume:

(B_1) $\sigma_{ij}(x)$, $b_i(x)$ $(1 \leqq i,\ j \leqq m)$ are uniformly Lipschitz continuous. If $1 \leqq h \leqq k_0$ then $\sigma_{ij}(z_h) = 0$, $b_i(z_h) = 0$. If $k_0 + 1 \leqq h \leqq k$,

$$\sum_{i\,j=1}^{m} a_{ij}\nu_i\nu_j = 0 \quad \text{on } \partial G_h, \tag{2.1}$$

$$(b,\nu) + \tfrac{1}{2}\sum_{i\,j=1}^{m} a_{ij}\frac{\partial^2\rho_h}{\partial x_i\,\partial x_j} = 0 \quad \text{on } \partial G_h, \tag{2.2}$$

where $\rho_h(x) = \text{dist.}(x,\partial G_h)$ $(x \notin \text{int}\, G_h)$, ν is the outward normal to ∂G_h, and $a_{ij} = \sum_{r=1}^{m}\sigma_{ir}\sigma_{jr}$.

Let $\tilde{G} = R^m - \bigcup_{j=1}^{k} G_j$. If σ_{ij} are C^1 functions in a neighborhood of ∂G_h and if (2.1) holds, then (2.2) is equivalent to

$$\sum_{i=1}^{m}\left\{b_i - \tfrac{1}{2}\sum_{j=1}^{m}\frac{\partial a_{ij}}{\partial x_j}\right\}\nu_i = 0; \tag{2.3}$$

that is, ∂G_h belongs to the part Σ_{12} of the boundary of $\tilde{G}$ (by the notation of [7]).

Let $R(x)$ be a C^2 function in $\tilde{G}$, coinciding with $\rho_h(x)$ in a $\tilde{G}$-neighborhood of ∂G_h, coinciding with $|x|$ near ∞ and positive elsewhere. Let

$$\mathscr{A} = \sum_{i,j=1}^{m} a_{ij}\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j},$$

$$\mathscr{B} = \sum_{i=1}^{m} b_i\frac{\partial R}{\partial x_i} + \tfrac{1}{2}\sum_{i,j=1}^{m} a_{ij}\frac{\partial^2 R}{\partial x_i \partial x_j},$$

$$Q = \frac{1}{R}\left(\mathscr{B} - \frac{\mathscr{A}}{2R}\right). \tag{2.4}$$

We assume:

(B_2) For some $\eta > 0$ sufficiently small,

$$Q(x) < 0 \text{ if } \rho_h(x) < \eta \ (1 \leqq h \leqq k) \text{ or if } |x| > \frac{1}{\eta}. \tag{2.5}$$

(B_3) If $\eta \leqq R(x) \leqq 1/\eta$,

$$\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial R(x)}{\partial x_i}\frac{\partial R(x)}{\partial x_j} > 0 \text{ if } \nabla_x R(x) \neq 0, \tag{2.6}$$

$$\text{and} \quad \sum_{i,j=1}^{m} a_{ij}\frac{\partial^2 R(x)}{\partial x_i \partial x_j} > 0 \quad \text{if } \nabla_x R(x) = 0.$$

(B_4) The functions

$$\frac{\partial a_{ij}}{\partial x_i}, \ \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}, \ \frac{\partial b_i}{\partial x_i}$$

are locally Hölder continuous, and

$$\sum_{i,j}\left|\frac{\partial a_{ij}}{\partial x_i}\right| \leqq C, \quad \sum_{i,j}\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} - \sum_i \frac{\partial b_i}{\partial x_i} \leqq C \ (C \text{ constant}).$$

Consider the Cauchy problem (1.1), (1.2) and suppose:

(B_5) $f(x)$ is continuous, $|f(x)| \leqq C(1 + |x|^{\alpha})$ for some positive constants C, α and

$$f = \text{const.} = f_i \text{ on } \partial G_i \ (k_0 + 1 \leqq i \leqq k).$$

Let $f_i = f(z_i)$ for $1 \leqq i \leqq k_0$.

THEOREM 3. *Suppose* (B_1)–(B_5) *hold, and denote by* $u(t,x)$ *the solution of the Cauchy problem* (1.1), (1.2). *If* $x \in \hat{G}$, *then* $u(t,x)$ *is independent of the restriction of* f *to* $\bigcup_{h=k_0+1}^{k} G_h$, *and*

$$\lim_{t\to\infty} u(t,x) = \sum_{i=1}^{k} f_i p_i(x) \qquad p_i(x) \geqq 0, \ \sum_{i=1}^{k} p_i(x) = 1$$

where $p_i(x)$ *is the probability that* dist.$(\xi(t), \partial G_i) \to 0$ *as* $t \to \infty$, *when* $\xi(0) = x$.

The $p_i(x)$ are weak solutions of $Lu = 0$ in $\tilde{G}$.

The theorem is a consequence of the following result (Friedman and Pinsky [4]): Under the assumptions of Theorem 3, a.s. $\xi(t) \in \tilde{G}$ for all $t > 0$ and

$$P\{\lim_{t\to\infty} R(\xi(t)) = 0\} = 1 .$$

EXTENSION. Theorem 3 extends to the case where $\rho_h(x)$, for each $k_0 + 1 \leqq h \leqq k$, is a C^1 function that is piecewise C^2. This is the case when G_h is a convex body with piecewise C^3 boundary, or when G_h is a surface of dimension $\leqq m - 1$ with C^3 boundary.

3. Degenerate parabolic equations with $m = 2$

For $m = 2$ the angular behavior of $\xi(t)$ can be studied. This should lead to more precise results than in Theorem 3 whereby either the stability condition (2.5) is weakened or the restriction $f =$ const. on G_i $(k_0 + 1 \leqq i \leqq k)$ is removed. We shall give here some results in the special case corresponding to linear stochastic equations, i.e.,

$$\sigma_{ij}(x) = \sum_{l=1}^{2} \sigma_{ij}^l x_l, \qquad b_i(x) = \sum_{l=1}^{2} b_i^l x_l .$$

Thus, $k_0 = k = 1$ and $G = G_1 = \{0\}$.

Let

$$\begin{aligned} &\lambda = (\cos\phi, \sin\phi), \ \lambda^{\perp} = (-\sin\phi, \cos\phi), \\ &\sigma_j \ = (\sigma_{ij}^l), \ B = (b_i^l), \\ &\tilde{\sigma}(\phi) \ = \{ \textstyle\sum_{j=1}^{2} \langle \sigma_j\lambda, \lambda^{\perp}\rangle^2\}^{\frac{1}{2}} \\ &\bar{b}(\phi) \ = \langle B\lambda, \lambda^{\perp}\rangle - \langle a(\lambda)\lambda, \lambda^{\perp}\rangle . \end{aligned}$$

Assume:

(C_1) Either (i) $\tilde{\sigma}(\phi) \neq 0$ for all ϕ, and $\int_0^{2\pi} (b(\phi)/\tilde{\sigma}^2(\phi)) d\phi \neq 0$; or (ii) $\tilde{\sigma}(\phi) \not\equiv 0$ $\tilde{\sigma}(\phi)$ is not everywhere positive; then $\tilde{\sigma}(\phi)$ has precisely two (not necessarily distinct) zeros ϕ_1, ϕ_2 in the interval $[0, \pi)$; we require that either $\langle B\lambda_i, \lambda_i^{\perp}\rangle > 0$ for $i = 1, 2$, or $\langle B\lambda_i, \lambda_i^{\perp}\rangle < 0$ for $i = 1, 2$, where $\lambda_i = (\cos\phi_i, \sin\phi_i)$, $\lambda_i^{\perp} = (-\sin\phi_i, \cos\phi_i)$.

Let $f(x) = g(\phi)$ where $x = (r\cos\phi, r\sin\phi)$. The following theorem is proved in Friedman and Pinsky [3]:

THEOREM 4. *If* (C_1) *holds then* $\lim_{t\to\infty} u(t,x)$ *exists for any* x; *for* $x \neq 0$,

$$\lim_{t\to\infty} u(t,x) = E\int_0^{T_1} g(\phi(t))dt$$

where $\phi(t)$ *is the solution of*

$$d\phi = \tilde{b}(\phi)dt + \tilde{\sigma}(\phi)dw,$$

and T_1 *is the time it takes* $\phi(t)$ *to change by* 2π.

If

$$Q(\lambda) \equiv \langle B\lambda, \lambda\rangle + \tfrac{1}{2}\operatorname{tr} a(\lambda) - \langle a(\lambda)\lambda, \lambda\rangle > 0$$

or all λ, then the theorem remains true for $f(x) = g(r, \phi)$ where $g(r, \phi) \to g(\phi)$ as $r \to \infty$, uniformly with respect to ϕ.

The proof of Theorem 4 is based on analyzing the Laplace transform $\int_0^\infty e^{-st}u(t,x)dt$, using the strong Markov property and the facts that T_1 has finite expectation and that it is not it concentrated on a lattice (i.e., $E\exp(i\lambda T_1) \neq 1$ for any real $\lambda \neq 0$). The tauberian theorem of Landau-Ikehara is applied.

Assume, now, instead of (C_1):

(C_2) $\tilde{\sigma}(\phi) = 0$ for $\phi = \phi_1$, ϕ_2 where $0 \leqq \phi_1 < \phi_2 < \pi$, and $\tilde{\sigma}(\phi) \neq 0$ in (ϕ_1, ϕ_2); further, $\langle B\lambda_1, \lambda_1^{\perp}\rangle > 0$, $\langle B\lambda_2, \lambda_2^{\perp}\rangle < 0$.

This implies (see [3]) that, for x in the sector $\phi_1 < \phi < \phi_2$, the solution $u(t,x)$ depends only on the initial data f in the same sector.

Let $\phi_1 < \phi_0 < \bar{\phi} < \phi_2$ and denote by $T_{\phi_0\bar{\phi}}$ the least time it takes $\phi(t)$ to go from ϕ_0 to ϕ_0 after intersecting $\phi = \bar{\phi}$. By a proof similar to that of Theorem 4 we have:

THEOREM 4′. *If* (C_2) *holds and* $x = (r_0\cos\phi_0,\ r_0\sin\phi_0)$, *then* $\lim_{t\to\infty} u(t,x)$ *exists and*

$$\lim_{t\to\infty} u(t,x) = E\int_0^{T_{\phi_0\bar{\phi}}} g(\phi(t))dt.$$

If $Q(\lambda) > 0$ for all λ, then the theorem remains true for $f(x) = g(r,\phi)$, provided $g(r,\phi) \to g(\phi)$ as $r \to \infty$, uniformly with respect to ϕ.

4. Degenerate elliptic equations in the plane

The probabilistic methods developed in [3], [4] (which yield Theorems 3, 4) can also be applied to obtain new results for the Dirichlet problem for degenerate L, in case $m = 2$. Consider the elliptic equation

$$Lu = 0 \text{ in } D \quad (m = 2) \tag{4.1}$$

where D is a bounded domain. Assume that the boundary of D consists of three parts: $\Sigma_3, \Sigma_2, \Sigma_1$ as in Kohn and Nirenberg [7]. Suppose further that $\Sigma_1 = \Sigma_{11} \times \Sigma_{12}$ where $Q(x) < 0$ near Σ_{12} ($Q(x)$ is defined in (2.4)), and $Q(x) > 0$ near Σ_{11}. We also assume that $\Sigma_2 \cup \Sigma_3$ is closed and disjoint from Σ_1.

Under suitable assumptions on $\sigma_{ij}(x)$, $b_i(x)$, we determine a finite number of distinguished boundary points $\zeta_1, \cdots, \zeta_p$ on Σ_{12}. A small D-neighborhood of ζ_i is divided into two domains N_i^+, N_i^- by a "transversal" curve initiating at ζ_i. We now consider the problem of solving (4.1) subject to the boundary conditions:

$$\begin{aligned} &u = f \text{ on } \Sigma_2 \cup \Sigma_3; \ u(x) \to f^+(\zeta_i) \text{ if } x \in N_i^+, x \to \zeta_i; \\ &u(x) \to f^-(\zeta_i) \text{ if } x \in N_i^-, \ x \to \zeta_i \quad (1 \leqq i \leqq p), \end{aligned} \tag{4.2}$$

where f and $f^+(\zeta_i)$, $f^-(\zeta_i)$ are given. It can be proved [5] that this Dirichlet problem has a unique solution in $C^2(D)$. One can allow L to degenerate (in a certain manner) in a small neighborhood of a finite number of curves lying in D.

In a recent paper, Stroock and Varadhan [8] considered the Dirichlet problem (in any number of dimensions) by probabilistic methods. They prove the existence and uniqueness of a solution taking prescribed boundary values on $\Sigma_2 \cup \Sigma_3$. They assume that when the term $c(x)u$ does not occur in Lu (as it is in our case) then $\sup_{x \in D} E_x(\tau) < \infty$ where τ is the exit time of $\xi(t)$. This latter condition is not satisfied in our case.

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